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FAST TRACK COMMUNICATION

On the ODE/IM correspondence for minimal models

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Abstract

Within the framework of the ODE/IM correspondence, we show that the minimal conformal field theories with $c < 1$ emerge naturally from the monodromy properties of certain families of ordinary differential equations.

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1. Introduction

This note is about a unifying programme, the ODE/IM correspondence, which links two-dimensional conformal field theories and integrable models to the spectral theory of ordinary differential equations. The first instance of this correspondence [1] was based on an identity between the transfer matrix eigenvalues of certain integrable models in their conformal limits [2, 3], and the spectral determinants [4, 5] of second-order ordinary differential equations. Since the initial results of [1] and then [6], the ODE/IM correspondence has been used in various branches of physics ranging from condensed matter [7] to PT-symmetric quantum mechanics [8], and from boundary conformal field theory [9] to the study of non-compact sigma models [10]. It has also been linked with the geometric Langlands correspondence [11]. A recent review containing many more references is [12].

Early examples of the correspondence concerned the ground states of the integrable lattice models, albeit with possibly twisted boundary conditions. In the conformal field theory setting this gave access to primary fields, but not to their descendants. However, in [13] Bazhanov, Lukyanov and Zamolodchikov conjectured that the descendant fields could be found through relatively-simple generalizations of the initial differential equation. The new equations were obtained by modifying the initial potential, which in general has a regular singularity at zero and an irregular singularity at infinity, by introducing a level-dependent number of additional regular singularities in the complex plane, subject to a zero-monodromy condition around these extra singularities, though not about the origin. In this short note we will show that the minimal $c < 1$ conformal field theories can equivalently be associated with ODEs governed by a trivial monodromy property in the *whole* complex plane, including the origin. A natural quantization condition on the coefficient of the regular (Fuchsian) singularity at the origin emerges, and since this coefficient is related to the Virasoro vacuum parameter p of [2, 3] this restricts the resulting conformal weights, so as to match precisely the Kac tables of the minimal models $\mathcal{M}_{\mathsf{ab}}$.

2. The $c < 1$ minimal models

We begin with the basic observation of [1, 6], that the Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x^{2M} - E) + \frac{l(l+1)}{x^2}\right)\psi(x, E, l) = 0 \tag{1}$$

is related to conformal field theory. The Stokes relations associated with (1) imply constraints on its eigenvalues $E \in \{E_i\}$, given suitable boundary conditions, which coincide with the Bethe Ansatz equations (BAEs) for the twisted six-vertex model in its conformal ($c = 1$) limit (see, for example, [12]). The same BAEs emerge from the study of $c \leqslant 1$ CFTs in the framework developed by Bazhanov, Lukyanov and Zamolodchikov in [2, 3]. In the notation used in [13], equation (1) encodes the primary field of a Virasoro module with central charge c, vacuum parameter p and highest weight Δ, where

$$c(M) = 1 - \frac{6M^2}{M+1}, \qquad p = \frac{2l+1}{4M+4}, \qquad \Delta(M, l) = \frac{(2l+1)^2 - 4M^2}{16(M+1)}. \tag{2}$$

In [14] it was observed that for $2M$ rational and suitable values of l, the solutions to (1) will all lie on a finite cover of the complex plane, and that this translates into a truncation of the fusion hierarchy [15] of the associated integrable model. This is of particular interest because, for such truncations, the central charges and field contents map to those of the minimal models with $c < 1$ (see, for example, the discussion in section 3 of [16]). However only the simplest case of $2M$ integer and $l(l+1) = 0$ was discussed explicitly in [14], in part because the general monodromies of solutions are hard to unravel in the presentation (1). One of our aims in this note is to show that a much simpler treatment is possible, from which the Kac tables of minimal model primary fields emerge in a very natural fashion.

Working backwards, we first note that, for any two coprime integers $\mathsf{a} < \mathsf{b}$, the ground state of the minimal model $\mathcal{M}_{\mathsf{ab}}$ is found by setting

$$M + 1 = \frac{\mathsf{b}}{\mathsf{a}}, \qquad l + \frac{1}{2} = \frac{1}{\mathsf{a}} \tag{3}$$

in (1). This corresponds to the central charge $c_{\mathsf{ab}} = 1 - \frac{6}{\mathsf{ab}}(\mathsf{b}-\mathsf{a})^2$ and (lowest-possible) conformal weight $\Delta = \frac{1}{4\mathsf{ab}}(1 - (\mathsf{b}-\mathsf{a})^2)$.

We now observe, as in section 6 of [17], that the $l(l+1)/x^2$ term in (1) can be eliminated, for this value of l, by the following transformation:

$$x = z^{\mathsf{a}/2}, \qquad \psi(x, E) = z^{\mathsf{a}/4-1/2}y(z, E). \tag{4}$$

With a further rescaling $z \to (2/\mathsf{a})^{2/\mathsf{b}}z$, (1) becomes

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + z^{\mathsf{a}-2}(z^{\mathsf{b}-\mathsf{a}} - \tilde{E})\right)y(z, \tilde{E}) = 0, \tag{5}$$

where

$$\tilde{E} = \left(\frac{\mathsf{a}}{2}\right)^{2-2\mathsf{a}/\mathsf{b}} E. \tag{6}$$

Note that the change of variable has replaced the singular generalized potential $P(x) = x^{2\mathsf{b}/\mathsf{a}-2} - E + (1/\mathsf{a}^2 - 1/4)x^{-2}$, defined on a multi-sheeted Riemann surface, by a simple polynomial $W(z) = z^{\mathsf{a}-2}(z^{\mathsf{b}-\mathsf{a}} - \tilde{E})$. In particular, any solution to (5) is automatically single-valued around $z = 0$, and the truncation of the fusion hierarchy as explained in section 4 of [14] is made much more transparent.

To see which other primary states in the original model might have similarly-trivial monodromy, we keep l real with $l + 1/2 > 0$, but otherwise arbitrary, and again perform the change of variable (4). The result is now

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\tilde{l}(\tilde{l}+1)}{z^2} + z^{\mathsf{a}-2}(z^{\mathsf{b}-\mathsf{a}} - \tilde{E})\right) y(z, \tilde{E}, \tilde{l}) = 0, \tag{7}$$

where

$$2\left(\tilde{l} + \tfrac{1}{2}\right) = \mathsf{a}\left(l + \tfrac{1}{2}\right). \tag{8}$$

The Fuchsian singularity in (7) at $z = 0$ means that the equation admits a pair of solutions which generally have the power series expansions

$$\chi_1(z) = z^{\lambda_1} \sum_{n=0}^{\infty} c_n z^n; \qquad \chi_2(z) = z^{\lambda_2} \sum_{n=0}^{\infty} d_n z^n, \tag{9}$$

where $\lambda_1 = \tilde{l} + 1$ and $\lambda_2 = -\tilde{l}$ are the two roots of the indicial equation

$$\lambda(\lambda - 1) - \tilde{l}(\tilde{l} + 1) = 0. \tag{10}$$

A general solution to (7) can be expressed as $y(z, \tilde{E}, \tilde{l}) = \sigma\chi_1(z) + \tau\chi_2(z)$, and we shall demand that the transformed ODE should be such that, for arbitrary $\tilde{E}$, the monodromy of $y(z, \tilde{E}, \tilde{l})$ around $z = 0$ is projectively trivial, which is to say that

$$y(\mathrm{e}^{2\pi \mathrm{i}} z) \propto y(z). \tag{11}$$

This condition ensures that the eigenvalues obtained by imposing the simultaneous decay of solutions in a pair of asymptotic directions at infinity are independent of the path of analytic continuation between these two directions. We shall show that the condition imposes the following constraints on $\tilde{l}$:

(i) $2\tilde{l} + 1$ is a positive integer;
(ii) The allowed values of $2\tilde{l} + 1$ are those integers which cannot be written as $\mathsf{a}s + \mathsf{b}t$ with s and t non-negative integers. In other words they form precisely the set of holes of the infinite sequence

$$\mathsf{a}s + \mathsf{b}t, \qquad s, t = 0, 1, 2, 3 \ldots. \tag{12}$$

We shall call the integers (12) 'representable' and denote the set of them by $\mathbb{R}_{\mathsf{ab}}$.

As a consequence we shall see that, as $\tilde{l}$ runs over its 'allowed' values, the rational numbers

$$\Delta_{\tilde{l}} = \Delta(M, l)|_{M=\mathsf{b}/\mathsf{a}-1, l=l(\tilde{l})} = \frac{(2\tilde{l}+1)^2 - (\mathsf{a}-\mathsf{b})^2}{4\mathsf{ab}} \tag{13}$$

precisely reproduce the set of conformal weights of the primary states lying in the Kac table of the minimal model $\mathcal{M}_{\mathsf{ab}}$. Figures 1 and 2 illustrate the story for the Ising and Yang–Lee cases.

To establish these claims we first note that the requirement that the general solution $y(z)$ be projectively trivial means that $\chi_1(z)$ and $\chi_2(z)$ must have the same monodromy, which implies that the two roots of the indicial equation must differ by an integer

$$\lambda_1 - \lambda_2 = 2\tilde{l} + 1 \in \mathbb{N}. \tag{14}$$

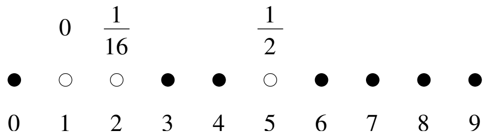

Figure 1. The holes (open circles) in the infinite sequence of integers defined in (12) for the critical Ising model $\mathcal{M}_{34}$. The holes are at 1, 2 and 5; the resulting conformal weights according to (13) are also shown, matching the primary field content of the Ising model.

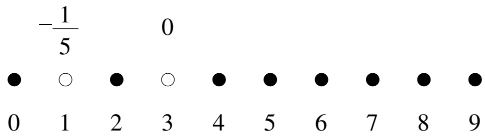

Figure 2. Holes for the Lee–Yang model $\mathcal{M}_{25}$, at 1 and 3. Notation as in figure 1.

This in turn restricts $\tilde{l}$ to be an integer or half integer, so that, naïvely, the allowed solutions are even or odd under a 2π rotation around $z = 0$. However it is well known that in such a circumstance, while $\chi_1(z)$ keeps its power series expansion (9), $\chi_2(z)$ generally acquires a logarithmic contribution:

$$\chi_2(z) = D\chi_1(z)\log(z) + \frac{1}{z^{\tilde{l}}}\sum_{n=0}^{\infty} d_n z^n. \tag{15}$$

Unless $D = 0$, this will spoil the projectively trivial monodromy of $y(z)$. We now show that $D = 0$ if and only if $2\tilde{l} + 1$ obeys the constraint (ii). In fact the logarithmic term is only absent when the recursion relations for d_n's with $D = 0$ admit a solution. These relations are

$$n(n - 2\tilde{l} - 1)d_n = d_{n-\mathsf{b}} - \tilde{E}d_{n-\mathsf{a}} \tag{16}$$

with the initial conditions $d_0 = 1$, $d_{m<0} = 0$.

Consider first the situation when $2\tilde{l} + 1 \notin \mathbb{R}_{\mathsf{ab}}$. Then, starting from the given initial conditions, the recursion relation (16) generates a solution of the form

$$\chi_2(z) = \frac{1}{z^{\tilde{l}}}\sum_{n=0}^{\infty} d_n z^n, \tag{17}$$

where the only nonzero d_n's are those for which the label n lies in the set $\mathbb{R}_{\mathsf{ab}}$. Given that $2\tilde{l} + 1 \notin \mathbb{R}_{\mathsf{ab}}$, for these values of n the factor $n(n - 2\tilde{l} - 1)$ on the lhs of (16) is never zero, and hence this procedure is well defined.

If instead $2\tilde{l} + 1 \in \mathbb{R}_{\mathsf{ab}}$, then equation (16) taken at $n = 2\tilde{l} + 1$ yields the additional condition

$$\tilde{E}d_{2\tilde{l}+1-\mathsf{a}} - d_{2\tilde{l}+1-\mathsf{b}} = 0, \tag{18}$$

which is inconsistent for generic $\tilde{E}$, and so the logarithmic term is required[5].

[5] The reader might wonder whether the lhs of (18) could vanish identically as a result of an exceptional cancellation between the two terms. This can be ruled out by observing that the monodromy property in z is unaltered by the change of variable $z \to \beta^{1/\mathsf{b}}z$. This rescaling leads to a more general version of (16): $n(n - 2\tilde{l} - 1)d_n = \alpha d_{n-\mathsf{a}} + \beta d_{n-\mathsf{b}}$ with $\alpha = -\tilde{E}\beta^{\mathsf{a}/\mathsf{b}}$. Then for the particular choice $\alpha = (-1)^{\mathsf{a}+1}$, $\beta = (-1)^{\mathsf{b}+1}$, one can see that $\alpha d_{n-\mathsf{a}}$ and $\beta d_{n-\mathsf{b}}$ must have the same sign for $n \leqslant 2\tilde{l} + 1$; hence, they cannot cancel identically.

Given the characterization (12) of $\mathbb{R}_{\mathsf{ab}}$, the set $\mathbb{Z}^+$ of non-negative integers can be written as a disjoint union

$$\mathbb{Z}^+ = \mathbb{R}_{\mathsf{ab}} \cup \mathbb{N}_{\mathsf{ab}}, \tag{19}$$

where $\mathbb{N}_{\mathsf{ab}}$ is the set of 'nonrepresentable' integers. If the coprime integers a and b are larger than 1 then $\mathbb{N}_{\mathsf{ab}}$ is non-empty; in fact $|\mathbb{N}_{\mathsf{ab}}| = \frac{1}{2}(\mathsf{a}-1)(\mathsf{b}-1)$, a result which goes back to Sylvester [18].

To characterize $\mathbb{N}_{\mathsf{ab}}$ more precisely, we start with the fact that given two coprime integers a and b, any integer n can be written as

$$n = \mathsf{a}s_0 + \mathsf{b}t_0, \qquad s_0, t_0 \in \mathbb{Z}. \tag{20}$$

This is a classical result of number theory. An intuitive proof uses Euclid's algorithm for the greatest common divisor of two integers. Alternatively one can invoke the Euler totient function φ[6] and the following theorem (see, for example, [19])

$$\mathsf{a}^{\varphi(\mathsf{b})} \equiv 1 (\mathrm{mod}\ \mathsf{b}), \tag{21}$$

which implies that $\mathsf{a}^{\varphi(\mathsf{b})} = 1 + h\mathsf{b}$ for some integer h. This immediately yields the solution $s_0 = n\mathsf{a}^{\varphi(\mathsf{b})-1}$, $t_0 = -nh$. However, for any given n, this is not the only possibility. More precisely, we have $\mathsf{a}s_0 + \mathsf{b}t_0 = \mathsf{a}s + \mathsf{b}t$ for some other pair of integers (s, t) if and only if $s = s_0 + \mathsf{b}k$, $t = t_0 - \mathsf{a}k$ for some $k \in \mathbb{Z}$. This is easily proved: rearrange (20) as

$$\mathsf{a}(s_0 - s) = \mathsf{b}(t - t_0). \tag{22}$$

Since $(\mathsf{a}, \mathsf{b}) = 1$, b must be a factor of $s_0 - s$ and so $s_0 - s = -\mathsf{b}k$ for some k. Dividing (22) by b then shows that $t_0 - t = \mathsf{a}k$, as required. Hence the possible representatives for each integer n constitute the line of points $(s, t) = (s_0 + \mathsf{b}k, t_0 - \mathsf{a}k)$, $k \in \mathbb{Z}$. For n to be a *positive* integer, $\mathsf{a}s + \mathsf{b}t > 0$, or $s > -\mathsf{b}t/\mathsf{a}$. If none of these points has both coordinates non-negative, then the corresponding n will be in $\mathbb{N}_{\mathsf{ab}}$. To keep t non-negative while making s as large as possible, we shift t by a multiple of a so that $0 \leqslant t < \mathsf{a}$. If s is still negative, then n will be in $\mathbb{N}_{\mathsf{ab}}$. The numbers we want are therefore represented by the points

$$\{(s, t), 0 \leqslant t < \mathsf{a}, -\mathsf{b}t/\mathsf{a} < s \leqslant -1\}. \tag{23}$$

Negating t, the allowed (trivial monodromy) values of $2\tilde{l} + 1$ are therefore

$$2\tilde{l} + 1 = \mathsf{a}s - \mathsf{b}t, \qquad 0 \leqslant t < \mathsf{a}, 1 \leqslant s < \mathsf{b}t/\mathsf{a}. \tag{24}$$

Figure 3 illustrates the argument.

Substituting back using (8) and (2), the allowed values for the conformal weights Δ precisely reproduce the Kac table for the minimal model $\mathcal{M}_{\mathsf{ab}}$:

$$\Delta = \Delta_{s,t} = \frac{(\mathsf{a}s - \mathsf{b}t)^2 - (\mathsf{a} - \mathsf{b})^2}{4\mathsf{ab}}, \qquad 1 \leqslant t < \mathsf{a}, 1 \leqslant s < \mathsf{b}t/\mathsf{a}. \tag{25}$$

It is striking that the full Kac table should emerge from such a simple consideration of the monodromy properties of the transformed differential equation (7). Finally, note that everything is symmetric in a and b so the same result can be obtained by starting from $M + 1 = \mathsf{a}/\mathsf{b}$ instead.

Another way to characterize $\mathbb{N}_{\mathsf{ab}}$ is through the generating function

$$\mathsf{P}(z) = \frac{(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}}) - (1 - z)(1 - z^{\mathsf{ab}})}{(1 - z)(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}})}. \tag{26}$$

[6] $\varphi(\mathsf{b})$ denotes the number of coprimes with b in the set $1, 2, \ldots, \mathsf{b}$.

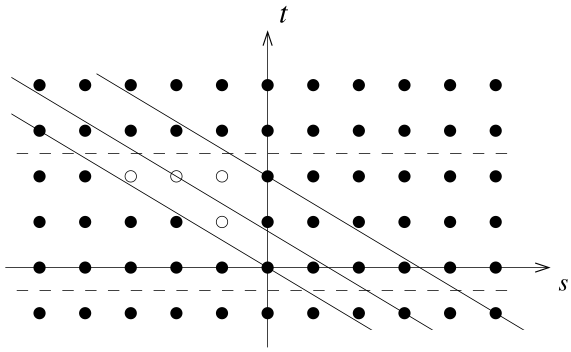

Figure 3. A graphical representation of the nonrepresentable integers for $\mathsf{a} = 3, \mathsf{b} = 5, \mathbb{N}_{\mathsf{ab}} = \{1, 2, 4, 7\}$. The elements of $\mathbb{N}_{\mathsf{ab}}$ correspond to the four unshaded points. Three of the lines $(s, t) = (s_0 + \mathsf{b}k, t_0 - \mathsf{a}k)$ have also been shown; each such line contains exactly one point in the region $0 \leqslant t < \mathsf{a}$ between the two dotted horizontal lines.

It is straightforward to show that such a rational function is actually a polynomial, because all the zeros of the denominator are cancelled by zeros of the numerator; we would like to show that

$$\mathsf{P}(z) = \sum_{2\tilde{l}+1\in\mathbb{N}_{\mathsf{ab}}} z^{2\tilde{l}+1}. \tag{27}$$

To see this, first note that Taylor expanding $1/(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}})$ yields

$$\frac{1}{(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}})} = \sum_{n\in\mathbb{R}_{\mathsf{ab}}} c_n z^n, \qquad (c_n > 0). \tag{28}$$

In order to get rid of the unknown c_n's, we combine the trivial identity

$$\frac{1}{1 - z^{\mathsf{a}}} = \sum_{n=0}^{\infty} z^{n\mathsf{ab}} \sum_{r=0}^{\mathsf{b}-1} z^{r\mathsf{a}} \tag{29}$$

with the similar one for $1/(1 - z^{\mathsf{b}})$ to write

$$\frac{1}{(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}})} = \sum_{n=0}^{\infty}(n + 1)z^{n\mathsf{ab}} \sum_{r=0}^{\mathsf{b}-1}\sum_{s=0}^{\mathsf{a}-1} z^{r\mathsf{a}+s\mathsf{b}}. \tag{30}$$

As a consequence the generating function of representable integers is

$$\frac{1 - z^{\mathsf{ab}}}{(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}})} = \sum_{n=0}^{\infty} z^{n\mathsf{ab}} \sum_{r=0}^{\mathsf{b}-1}\sum_{s=0}^{\mathsf{a}-1} z^{r\mathsf{a}+s\mathsf{b}} = \sum_{n\in\mathbb{R}_{\mathsf{ab}}} z^n. \tag{31}$$

Therefore the difference

$$\mathsf{P}(z) \equiv \frac{1}{1 - z} - \frac{1 - z^{\mathsf{ab}}}{(1 - z^{\mathsf{a}})(1 - z^{\mathsf{b}})} \tag{32}$$

is the sought after formula (27).

Before concluding this section we would like to mention that there is another ODE that can be associated with the same series of minimal models. This is the so-called $A_2^{(2)}$ description, related to ϕ_{12}, ϕ_{21} and ϕ_{15} perturbations [20]. After a simple change of variable, the relevant third-order ϕ_{12}-related ODE can be cast into the form

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}z}-\frac{\tilde{g}}{z}\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}+\frac{\tilde{g}}{z}\right)+(z^{2\mathsf{b}-3}-\tilde{E}z^{\mathsf{a}-3})\right]y(z,E,\tilde{g})=0. \tag{33}$$

Swapping a and b gives the ϕ_{21}-related ODE, while replacing a with $2\mathsf{a}$ and b with $\mathsf{b}/2$ yields the ϕ_{15} equation. In all cases, the indicial equation is

$$(\lambda+\tilde{g})(\lambda-1)(\lambda-(\tilde{g}+2))=0. \tag{34}$$

The zero relative monodromy condition among the three solutions to (33) requires $(\tilde{g}+1)$ to be a (positive) integer, and to avoid logarithmic terms we should also simultaneously impose the following two conditions:

$$(\tilde{g}+1)\notin\{2\mathsf{b}t+\mathsf{a}s\}, \qquad 2(\tilde{g}+1)\notin\{2\mathsf{b}t+\mathsf{a}s\} \tag{35}$$

with $s,t=0,1,2,\ldots$. For a odd, it is easy to check that equations (35) lead to the same set of integers as the $su(2)$-related case discussed above, while only a subset is recovered for a even. For the ϕ_{21}-related case, the opposite situation occurs: for a even the full Kac table is recovered, while for a odd only a subset is found.

3. Further generalizations and conclusions

There are many possible generalizations of the above results. The existence of a simpler version, equation (7), of the basic ODE for minimal CFTs is not restricted to $c<1$ Virasoro models, but generalizes to the higher $su(2)$ coset CFTs of [21] and to the ABCD-related theories of [22]. The pseudo-differential equations listed in section 3 of [22] include the minimal models

$$\frac{\hat{\mathfrak{g}}_L\times\hat{\mathfrak{g}}_K}{\hat{\mathfrak{g}}_{L+K}}, \qquad \mathfrak{g}=A_n,B_n,C_n,D_n \tag{36}$$

at fractional level $L=K\mathsf{a}/(\mathsf{b}-\mathsf{a})-h^\vee$ with $\mathsf{b}-\mathsf{a}=Ku$, and $u=1,2,\ldots$ (using the notation of appendix 18.B of [23]). We checked that after simple changes of variable, these equations reduce to equations similar in form to the originals, but for a change in the 'potential', as follows:

$$P_K(x)=(x^{h^\vee(\mathsf{b}-\mathsf{a})/\mathsf{a}K}-E)^K \longrightarrow W_{(K,L)}(z)=z^{\mathsf{a}-h^\vee}(z^{(\mathsf{b}-\mathsf{a})/K}-\tilde{E})^K. \tag{37}$$

It is striking that when both L and K are integers the CFT is unitary and $W_{(K,L)}(z)$ simplifies further to

$$W_{(K,L)}(z)=z^L(z-\tilde{E})^K. \tag{38}$$

Equation (38) motivates some simple comments and speculations. First we observe that the $K\leftrightarrow L$ invariance of (36) manifests itself in (38) as a shift in z. As a consequence of this symmetry, lateral quantization problems for the ground-state ODEs and the associated Stokes multipliers are, up to $\tilde{E}\to-\tilde{E}$, invariant under the exchange of L and K. However, this symmetry is explicitly broken in equations with extra Fuchsian singularities as in (7). A possible remedy is to treat the points $z=0$ and $z^{(\mathsf{b}-\mathsf{a})/K}=\tilde{E}$ more democratically. For instance, in unitary models the symmetry is globally restored after the addition of a second singularity at $z=\tilde{E}$, ensuring that the set of ODEs for the primary fields in a given CFT is mapped into itself by the transformation. We suspect that a similar modification may also resolve the problem of the missing states in the $A_2^{(2)}$ example discussed at the end of section 2.

A further possibility is suggested by equation (38). Consider the following multi-parameter generalization of (38):

$$W(z, \mathbf{e}) = z \prod_{i=1}^{K+L-1} (z - e_i), \tag{39}$$

where the constants e_i $(i = 1, 2 \ldots, K + L - 1)$ are free parameters. To keep the discussion brief, we shall restrict attention to the ground-state equation for $L + K = 3$ and $\mathfrak{g} = su(2)$. Then if $e_1 = 0$ and $e_2 = \tilde{E}$ the corresponding ODE is related to the tricritical Ising model $\mathcal{M}_{45}$, while for $(e_1, e_2) = (\pm\sqrt{\tilde{E}}, \mp\sqrt{\tilde{E}})$ the equation corresponds to $\mathcal{M}_{35}$. Therefore this simple two-parameter model interpolates smoothly between equations associated with $\mathcal{M}_{45}$ and $\mathcal{M}_{35}$. This phenomenon has a counterpart in the homogeneous sine-Gordon model corresponding to integrable perturbations of the $\hat{su}(3)_2/U(1)^2$ coset model. The thermodynamic Bethe ansatz (TBA) equations for this model [24, 25] have two independent scale parameters μ_1 and μ_2. If one of these parameters is set to zero, the TBA equations reduce to those for $\mathcal{M}_{45} + \phi_{13}$ [26], while for $\mu_1 = \mu_2$ the TBA equations map into a pair of identical equations for $\mathcal{M}_{35} + \phi_{13}$ [27]. This, and other simple considerations, suggest that ODEs with multi-parameter potentials of the form (39) may have an interesting interpretation in terms of conformal field theory. Much more work will be needed in order to give this observation a more solid grounding, but we feel that it will be an interesting direction for future exploration.

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